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~~(Presented by Academician I. G. Petrovskiy, April 13, 1954)~~.

Let us call some closed region G of an n-dimensional space toroidal, if it may be obtained from a topological torus or from the spherical neighborhood of a circle T by means of the exclusion of not more than a finite number of regions entirely embedded in T. A toroidal area which does not coincide with the torus is, for example, the space enclosed between two torus surfaces. To each point of a toroidal region may be ascribed an angular coordinate which determines the meridional surface upon which this point lies.

Definition 1.

We shall say that the trajectory $\varphi(P, t)$ of a dynamic system $\frac{dx_i}{dt} = f_i(x_1, x_2, \ldots, x_n)$, entirely embedded in G, determines an oscillatory regime with respect to G if: a) $\varphi(P, t)$ is stable according to Poisson; b) there exists such a number $\tau_0 > 0$, that during the time interval $\tau_q \leqslant \tau_0$ the point $q = \varphi(P, t)$ intersects each meridional surface once and only once.

The regime described by a trajectory satisfying b) and not satisfying condition a) will be called asymptotic to the oscillatory regime.

For the Goddard Space Fligh Center, 1966.

Definition 2

We shall call the function

$$V(x_1, x_2, \ldots, x_n) = \frac{F_1(x_1, x_2, \ldots, x_n)}{F_2(x_1, x_2, \ldots, x_n)}$$

the Lyapunov rotational function with respect to the closed function G, if the following conditions are satisfied: F_1 and F_2 are continuously differentiable functions in G; 2) equation $F_1 - CF_2 = 0$, $-\infty \leqslant C \leqslant +\infty$, constitutes the equation of a bundle of surfaces, the axis of which does not pass through the points of G, the surfaces $V=+\infty$ & $V=-\infty$ coinciding with the surface $F_2(x_1, x_2, \ldots, x_n)=0$; (3); each surface of the bundle $F_1 - CF_2 = 0$ intersects G; 4) each point of $\overline{G}$ belongs to one of the surfaces of the bundle.

Let us consider a derivative of $V(x_1, x_2, \ldots, x_n)$, taken by virtue of the system of equations $dx_i/dt = f_i(x_1, x_2, \ldots, x_n)$, i.e., let us consider the expression

$$\frac{dV}{dt} = \frac{\sum_{i=1}^{n} \left(\frac{\partial F_1}{\partial x_i} F_2 - \frac{\partial F_2}{\partial x_i} F_1\right) f_i}{F_2^2(x_1, x_2, \ldots, x_n)}.$$

Taking into account that $F_1/F_2 = C$*, this expression may be rewritten as

$$\frac{dV}{dt} = \frac{\sum_{i=1}^{n} \left(\frac{\partial F_1}{\partial x_i} F_2 - \frac{\partial F_2}{\partial x_i} F_1\right) f_i}{F_1^2(x_1, x_2, \ldots, x_n)} C^2 = \Gamma(x_1, x_2, \ldots, x_n) C^2.$$

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* Along the integral curve, C may be regarded as a function of t.

In addition let us consider separately the value of dV/dt on the surface V=0, i.e. on the surface $F_1=0$. dV/dt has the following form

$$\left(\frac{dV}{dt}\right)_{C=0} = \frac{\sum_{i=1}^{n} \frac{\partial F_1}{\partial x_i} f_i}{F_2} = B(x_1, x_2, \ldots, x_n).$$

Using these designations, we formulate a theorem.

Theorem 1

"If all the trajectories of the system enter into the interior of G through all the boundary points of a toroidal closed region $\overline{G}$ that contains no singular points, and if with respect to $\overline{G}$ there exists a Lyapunov rotational function V = C, for which $|\Gamma| > a^2 > 0$ and $|B| > \alpha^2 > 0$ for the points of $\overline{G}$, then G contains an oscillatory regime."

Let us consider an arbitrary internal point P_0 of the region G, lying on the surface V = 0 when t = 0; since, under the condition of the theorem,

$$\left(\frac{dV}{dt}\right)_{C=0} > \alpha^2 > 0,$$

it follows that the phase point when t > 0 cannot remain on the surface V = 0 and will pass to surfaces corresponding to larger values of C; since for any value of C that differs from zery

$$\left(\frac{dV}{dt}\right)_C > a^2 C^2,$$

the trajectory will pass to higher values of C when t increases, i.e., along the trajectory C may be considered as a monotonic function of t and, consequently, along the trajectory $\frac{dV}{dt}=\frac{dC}{dt}$. We shall show that after a finite time interval the trajectory will emerge to the surface $C=+\infty$.

We shall find this time interval from the equality

$$T=\int_0^T dt=\int_0^\infty \frac{dC}{dV/dt}.$$

We have

$$T=\int_0^\delta \frac{dC}{dV/dt}+\int_\delta^\infty \frac{dC}{dV/dt},$$

where δ is so small that for $0<C<\delta$ $dV/dt>\gamma_\delta>0$; we have

$$\int_0^\delta \frac{dC}{dV/dt}\leqslant\frac{\delta}{\gamma_\delta}.$$

Let us evaluate the second integral:

$$\int_\delta^\infty \frac{dC}{dV/dt}=\int_\delta^\infty \frac{dC}{|\Gamma|C^2}<\frac{1}{a^2}\int_\delta^\infty \frac{dC}{C^2}=\frac{1}{a^2}\left[-\frac{1}{C}\right]_\delta^\infty=\frac{1}{a^2\delta}.$$

Thus

$$T\leqslant\frac{\delta}{\gamma_\delta}+\frac{1}{a\delta^2}.$$

The same evaluation also takes place for intervals of the change of C from $+\infty$ to 0, from 0 to $-\infty$ and from $-\infty$ to 0.

If we now introduce the angular coordinate φ by means of the formula $\operatorname{tg}\varphi=C$, it is clear that the phase point will return to the initial surface if φ passes from 0 to 2π, i.e., C will pass from 0 to $+\infty$ from $-\infty$ to 0, from 0 to $-\infty$, and from $-\infty$

to 0.

Since the evaluation carried out by us is suitable for each of these intervals of change of C, after a time period of

$$\tau_0 \leqslant 4\left(\frac{\delta}{\gamma_8} + \frac{1}{a^2\delta}\right)$$

the phase point will return to the initial surface.

Thus each trajectory which begins in G on the surface C = 0 will describe either an oscillatory regime or a regime that is asymtotic to an oscillatory one. On the basis of the general theorems, we conclude that within $\overline{G}$ there will be a recurrent trajectory, apparently different from a singular point. This recurrent trajectory will already satisfy both conditions a) and b). The numbers D* and $2\pi/\tau_0$ will characterize the amplitude and the frequency of the obtained oscillatory regime.

Example

We are considering the system

$$\frac{dx}{dt} = y + \varepsilon x^2 P_1(x, y),$$
$$\frac{dy}{dt} = -x + \varepsilon x^2 Q_1(x, y),$$
$$\frac{dz}{dt} = -z + R(x, y),$$

where $P_1, Q_1, R(x, y)$ are continuous in the toroidal region $r^2 \leqslant x^2 + y^2 \leqslant R^2; 0 \leqslant z \leqslant a.$

If the following conditions are fulfilled:

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* Diameter of the region Cm.

$$\begin{array}{ll} 0 < R(x, y) < a & \text{для} \quad r^2 \leqslant x^2 + y^2 \leqslant R^2; \\ xP + yQ > 0 & \text{для} \quad x^2 + y^2 = r^2; \\ xP + yQ < 0 & \text{для} \quad x^2 + y^2 = R^2, \end{array}$$

then in a selected toroidal region there will be an oscillatory regime for those values of ε, for which the inequalities

$$-1 + \varepsilon x^2 Q < 0, \quad \frac{r^2}{R^2} + \varepsilon x^2 \frac{|xQ - yP|}{R^2} > 0,$$

are valid in the toroidal region. Along this path it is easy to construct examples of systems which possess a preassigned number of geometrically different oscillation regimes.

Definition 3

We shall call the closed set S a local cross section of the trajectory flux, if there exists such a number T, not depending upon the choice of the point S, that every arc $f(P, -T, +T)$ of the trajectory $\varphi(P, t)$, which emerges from S when t = 0, has only one point in common with S.

Using the Brauer theorem concerning the existence of a fixed transformation point, the following theorem may be established:

Theorem 2

"Let G be a toroidal region, let it have the cross section $S^{(n-1)}$, which is an element of the dimensionality n-1, and let this element be a local cross section of the trajectory flux passing through $S^{(n-1)}$. Then, if each trajectory $\varphi(P, t)$, which originates in the points of $S^{(n-1)}$ returns again to $S^{(n-1)}$ for some value t = w, then a periodic solution exists within G."

On the basis of theorems 2 and 1, it is easy to disclose the existence of a periodic solution for a dynamic system such as the example considered above.

It is of interest to note that on the basis of this principle an oscillatory egime was found in an article by Rauch (1). Essentially, the same considerations from the basis of some examples dealt with in the well-known study by A.A.Andronov, N.N. Bautin, and G.S. Gorelik (2).

Finally, let us remark that the theorems under consideration are not required for the investigation of systems on a phase plne, since on a plane the existence of an annular region, into which all the trajectories enter, is already sufficient for the existence of a periodic solution; whereas, as has been shown by an example of Fuller, the presence of toroidal area, into which enter all the trajectories, guarantees the existence neither of periodic solutions in this region, nor of an oscillatory regime in our sense.

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